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Dispersion of Thermoelastic Waves in a Plate With and Without Energy Dissipation

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In this paper, the dispersion and energy dissipation of thermoelastic plane harmonic waves in a thin plate bounded by insulated traction-free surfaces is studied on the basis of three generalized theories of thermoelasticity. The frequency equations corresponding to the symmetric and antisymmetric modes of vibration of the plate are obtained. Some limiting and particular cases of the frequency equations are then discussed. Results obtained in three theories of generalized thermoelasticity are compared. The results for the coupled thermoelasticity can be obtained as particular cases of the results by setting thermal relaxations times equal to zero. Numerical evaluations relating to the lower modes of the symmetric and antisymmetric waves are presented for an aluminum alloy plate.

KEY WORDS: energy dissipation; frequency; harmonic waves; plate; thermal relaxations; thermoelasticity.

1. INTRODUCTION

It is well known that the propagation of elastic waves in an infinite elastic plate with traction free surfaces is governed by the Rayleigh–Lamb frequency equation [1]. The classical Rayleigh–Lamb waves may be classified by their symmetry with respect to the middle plane. In the classical theory of thermoelasticity, when a homogeneous isotropic elastic solid is subjected to a thermal or mechanical disturbance, the effects in the temperature and

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displacement fields are felt at an infinite distance from the source of disturbance instantaneously. This implies that thermal waves propagate with an infinite velocity, which is physically impossible. During the last three decades, nonclassical theories have been developed, which are free from this paradox.

The thermoelastic theories proposed by Lord and Shulman [2] and Green and Lindsay [3] (here called the LS and GL theories, respectively) have aroused much interest in recent years. These theories are generalizations of the coupled thermoelasticity theory [4] and are formulated by using a form of the heat conduction equation that includes the time needed for acceleration of the heat flow. The LS theory introduces a single time constant to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate and the rate of change of heat generation. In the GL theory, on the other hand, thermal and thermomechanical relaxations are governed by two different time constants. These theories eliminate the paradox of infinite velocity of heat propagation. In the GL theory, the heat cannot propagate at a finite speed unless the stresses depend on the thermal wave velocity, whereas in LS theory the heat can propagate at a finite speed even though the stresses are independent of the thermal wave velocity. Thus, the physical interpretations as well as the assumptions of dynamic thermoelastic processes of the LS and GL theories are distinctively different. Many studies have explored this difference to quantify the implications of their differences in particular field problems, e.g., the reciprocity theorem, the uniqueness of a solution, and an energy law, for both theories.

Recently, the theory of thermoelasticity without energy dissipation, which provides sufficient basic modifications to the constitutive equation to permit treatment of a much wider class of flow problems, has been proposed by Green and Naghdi [5] (called the GN theory). The discussion presented in Ref. 5 includes the derivation of a complete set of governing equations of the linearized version of the theory for homogeneous and isotropic materials in terms of displacement and temperature fields and a proof of the uniqueness of the solution of the corresponding initial mixed boundary value problem. The uniqueness of the solution for an initial boundary value problem in this theory, formulated in terms of stress and energy flux, has been established in Ref. 6. Chandrasekharaiah and Srinath [7] investigated one-dimensional wave propagation in the context of the GN theory. Verma [8] studied the field equations of linear thermoelasticity in the GN theory with the help of integral transforms. They have discussed the dynamic behavior of an elastic half-space due to a thermal shock and a mechanical load on the boundary and found that the disturbances consist of two coupled waves that propagate with finite speeds, without attenuation.

Derisiewicz [9] considered the propagation of waves in thermoelastic plates under plane strain. The propagation of thermoelastic waves in a plate under plane stress by using generalized theories of thermoelasticity has been studied by Chandrasekharaiah and Srinath [10], and by Massalas [11]. We refer the reader to several reviews and papers [12–18] on this topic that have considered the propagation of generalized thermoelastic waves in plates of isotropic media under different boundary conditions. Massalas and Kalpakidis [19] used the LS theory to study the characteristics of wave motion in a thin plate under plane stress with mixed boundary conditions. They used Lamé's potentials to derive the frequency equation. Verma and Hasebe [20] studied the propagation of generalized thermoelastic vibrations in infinite plates using the LS and GN theories.

In this paper, we investigate the propagation of plane harmonic waves in an infinite homogeneous isotropic plate of thickness $2d$ using the GL and GN theories of generalized thermoelasticity. The frequency equations corresponding to the symmetric and antisymmetric modes of vibration of the plate are obtained; some limiting and particular cases of the frequency equations are then discussed. A comparison of the results for the GL, LS, and GN theories of generalized thermoelasticity is also presented. We found that in the GN theory, coupled waves propagate at finite speeds, without attenuation. It has also been observed that, on the whole, the results obtained with the GN theory are qualitatively similar to those with the GL and LS theories. Numerical evaluations relating to the lower modes of the symmetric and antisymmetric waves are presented for an aluminum alloy plate.

2. HARMONIC WAVES IN A THERMOELASTIC PLATE

We consider an infinite homogeneous isotropic thermally conducting elastic plate at uniform temperature T_0 in the undisturbed state having thickness $2d$. Let the faces of the plate be the planes $z = \pm d$, referring to a rectangular set of Cartesian axes $O(x, y, z)$. We choose the x axis to be the direction of propagation of waves so that all particles on a line parallel to the y axis are equally displaced. Therefore, all the field quantities will be independent of the y coordinate. The motion is supposed to take place in two dimensions (x, z). Here u and w are the displacements in the x and z directions, respectively. In the linear generalized theory of thermoelasticity, the governing field equations for the temperature $T(x, z, t)$ and the displacement vector $\mathbf{u}(x, z, t) = (u, 0, w)$ in the absence of body forces and heat sources are given by Ref. 3.

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)+(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x\,\partial z}\right)-\beta\frac{\partial}{\partial x}\left(T+\tau_1\frac{\partial T}{\partial t}\right)=\rho\frac{\partial^2 u}{\partial t^2}$$

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)+(\lambda+\mu)\left(\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 u}{\partial x\,\partial z}\right)-\beta\frac{\partial}{\partial z}\left(T+\tau_1\frac{\partial T}{\partial t}\right)=\rho\frac{\partial^2 w}{\partial t^2}$$

$$K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right)-\rho C_{\mathrm{e}}\left(\frac{\partial T}{\partial t}+\tau_0\frac{\partial^2 T}{\partial t^2}\right)=T_0\beta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)+\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right)\right] \tag{1}$$

where

$$\beta=(3\lambda+2\mu)\,a_{\mathrm{t}}$$

λ and μ are Lamé's parameters; ρ is the density of the medium; C_{e} is the specific heat at constant strain; τ_0 and τ_1 are the thermal and thermomechanical relaxation times, respectively; and K and α_{t} are, respectively, the coefficients of thermal conductivity and linear thermal expansion. The parameters τ_0 and τ_1 satisfy the inequality $\tau_1 \geqslant \tau_0 \geqslant 0$.

If $\tau_1 \neq 0$, the stresses depend on the thermal wave velocity, and if $\tau_0 \neq 0$, the heat propagates at a finite speed. Since $\tau_0 \neq 0$ implies $\tau_1 \neq 0$, it follows that the heat cannot propagate at a finite speed, unless the stresses depend on the thermal wave velocity.

We define the following dimensionless quantities:

$$\begin{gathered}
x^*=\frac{v_1}{k_1}x, \qquad z^*=\frac{v_1}{k_1}z, \qquad t^*=\frac{v_1^2}{k_1}t\\
u^*=\frac{v_1^3\rho}{k_1\beta T_0}u, \qquad w^*=\frac{v_1^3\rho}{k_1\beta T_0}w, \qquad T^*=\frac{T}{T_0}\\
\tau_0^*=\frac{v_1^2}{k_1}\tau_0, \qquad \tau_1^*=\frac{v_1^2}{k_1}\tau_1, \qquad c_2=\frac{\mu}{2(\lambda+2\mu)}\\
c_3=1-c_2, \qquad \varepsilon_1=\frac{\beta^2 T_0}{\rho C_{\mathrm{e}} v_1^2}
\end{gathered} \tag{2}$$

where $v_1=((\lambda+2\mu)/\rho)^{1/2}$ is the velocity of compressional waves and $k_1=K/(\rho C_{\mathrm{e}})$ is the thermal diffusivity in the x direction. Here ε_1 is the thermoelastic coupling constant, and τ_0^* and τ_1^* are the thermal relaxation constants. On substituting Eq. (2) into Eq. (1), after suppressing the $*$, we obtain

$$\frac{\partial^2 u}{\partial x^2} + c_2 \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} + c_3 \frac{\partial^2 w}{\partial x\,\partial z} = \frac{\partial}{\partial x}\left(T + \tau_1 \frac{\partial T}{\partial t}\right)$$

$$c_2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} + c_3 \frac{\partial^2 u}{\partial x\,\partial z} - \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial z}\left(T + \tau_1 \frac{\partial T}{\partial t}\right) \tag{3}$$

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) - \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right) = \varepsilon_1 \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right)\right]$$

The stresses relevant to our problem in the plate are

$$\tau_{zz} = \left[(1 - 2c_2)\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \left(T + \tau_1 \frac{\partial T}{\partial t}\right)\right] \beta T_0$$

$$\tau_{zx} = \beta T_0 c_2 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{4}$$

and $\partial T/\partial z$ is the temperature gradient.

For a plane harmonic wave traveling in the x direction, the solutions u, w, and T of Eq. (3) take the form

$$(u, w, T) = (f(z), g(z), h(z)) \exp[i\xi(x - ct)] \tag{5}$$

where c $(=\omega/\xi)$ and ξ are the phase velocity and wave number, respectively; ω is the circular frequency; and $i = \sqrt{-1}$. Substituting for u, w, and T from Eq. (5) into Eq. (3), we get

$$(c_2 D^2 - \xi^2 + \xi^2 c^2) f + i\xi c_3 Dg - \xi^2 c \tau_G h = 0$$

$$i\xi c_3 Df + (D^2 - c_2 \xi^2 + \xi^2 c^2) g - i\xi c \tau_G Dh = 0$$

$$\xi^2 \varepsilon_1 cf - i\xi c \varepsilon_1 Dg - (D^2 - \xi^2 + \tau \xi^2 c^2) h = 0$$

where

$$D = \frac{d}{dz}, \qquad \tau = \tau_0 + \frac{i}{\xi c}, \qquad \tau_G = \tau_1 + \frac{i}{\xi c}$$

The solution to Eqs. (6) is

$$\begin{aligned} f(z) &= P_1 \exp(-\xi s_1 z) + P_2 \exp(-\xi s_2 z) + P_3 \exp(-\xi s_3 z) \\ &\quad + Q_1 \exp(\xi s_1 z) + Q_2 \exp(\xi s_2 z) + Q_3 \exp(\xi s_3 z) \\ g(z) &= m_1 P_1 \exp(-\xi s_1 z) + m_2 P_2 \exp(-\xi s_2 z) + m_3 P_3 \exp(-\xi s_3 z) \\ &\quad - m_1 Q_1 \exp(\xi s_1 z) - m_2 Q_2 \exp(\xi s_2 z) - m_3 Q_3 \exp(\xi s_3 z) \\ h(z) &= \xi[l_1 P_1 \exp(-\xi s_1 z) + l_2 P_2 \exp(-\xi s_2 z) + l_3 P_3 \exp(-\xi s_3 z) \\ &\quad + l_1 Q_1 \exp(\xi s_1 z) - l_2 Q_2 \exp(\xi s_2 z) - l_3 Q_3 \exp(\xi s_3 z)] \end{aligned} \tag{7}$$

where

$$\begin{aligned} m_j &= i s_j, \qquad m_3 = 0 \\ l_j &= \frac{1}{i}[s_j^2 + c^2 - 1], \qquad l_3 = 0, \quad j = 1, 2 \end{aligned} \tag{8}$$

P_j and Q_j ($j = 1, 2, 3$) are arbitrary constants, and s_1^2 and s_2^2 are the roots of the equation

$$s^4 + As^2 + B = 0 \tag{9}$$

where

$$A = [(c^2 - 2) + (\tau c^2 + \varepsilon_1 \tau_{\mathbf{G}} c^2)]$$

$$B = [1 - (\tau c^2 + \varepsilon_1 \tau_{\mathbf{G}} c^2) + c^4 \tau - c^2)]$$

and

$$s_3^2 = 1 - \frac{c^2}{c_2} \tag{10}$$

s_1^2 and s_2^2 correspond to the longitudinal and thermal waves, whereas s_3^2 corresponds to the transverse wave. This is in agreement with the corresponding results obtained by Nayfeh and Nasser [24].

The displacements and temperature of the plate are thus

$$
\begin{aligned}
u &= [P_1 \exp(-\xi s_1 z) + P_2 \exp(-\xi s_2 z) + P_3 \exp(-\xi s_3 z) \\
&\quad + Q_1 \exp(\xi s_1 z) + Q_2 \exp(\xi s_2 z) + Q_3 \exp(\xi s_3 z)] \exp[i\xi(x - ct)] \\
w &= [m_1 P_1 \exp(-\xi s_1 z) + m_2 P_2 \exp(-\xi s_2 z) + m_3 P_3 \exp(-\xi s_3 z) \\
&\quad - m_1 Q_1 \exp(\xi s_1 z) - m_2 Q_2 \exp(\xi s_2 z) \\
&\quad - m_3 Q_3 \exp(\xi s_3 z)] \exp[i\xi(x - ct)] \\
T &= \xi[l_1 P_1 \exp(-\xi s_1 z) + l_2 P_2 \exp(-\xi s_2 z) + l_3 P_3 \exp(-\xi s_3 z) \\
&\quad + l_1 Q_1 \exp(\xi s_1 z) + l_2 Q_2 \exp(\xi s_2 z) + l_3 Q_3 \exp(\xi s_3 z)] \exp[i\xi(x - ct)]
\end{aligned}
\tag{11}
$$

3. FREQUENCY EQUATIONS AND THERMOELASTIC DISSIPATION

The boundary conditions are that stresses and the temperature gradient on the surfaces of the plate should vanish. Hence, for all x and t,

$$\tau_{zz} = \tau_{xz} = T_{,z} = 0 \qquad \text{on} \quad z = \pm d \tag{12}$$

Substituting the expression, Eq. (11), for the displacement components and temperature into Eq. (4), we obtain the stresses and the temperature gradient. Substituting the expressions for the stresses and the temperature gradient, Eq. (12), we obtain six equations involving the arbitrary constants P_1, P_2, P_3, Q_1, Q_2, and Q_3:

$$
\begin{aligned}
&\sum_{j=1}^{3} (iF - m_j s_j - l_j)(P_j e^{-\xi s_j d} + Q_j e^{\xi s_j d}) = 0 \\
&\sum_{j=1}^{3} (im_j - s_j)(P_j e^{-\xi s_j d} - Q_j e^{\xi s_j d}) = 0 \\
&\sum_{j=1}^{3} (-l_j s_j)(P_j e^{-\xi s_j d} - Q_j e^{\xi s_j d}) = 0 \\
&\sum_{j=1}^{3} (iF - m_j s_j - l_j)(P_j e^{\xi s_j d} + Q_j e^{-\xi s_j d}) = 0 \\
&\sum_{j=1}^{3} (im_j - s_j)(P_j e^{\xi s_j d} - Q_j e^{-\xi s_j d}) = 0 \\
&\sum_{j=1}^{3} (-l_j s_j)(P_j e^{\xi s_j d} - Q_j e^{-\xi s_j d}) = 0
\end{aligned}
\tag{13}
$$

where

$$F = 1 - 2c_2$$

For the six boundary conditions to be satisfied simultaneously, the determinant of the coefficients of the arbitrary constants must vanish. This gives an equation for the frequency of the plate oscillations. The frequency equation is found to factorize into two factors, each of which yields the equations,

$$\begin{aligned} &D_1 G_1 \coth(\xi s_1 d) - D_2 G_2 \coth(\xi s_2 d) + D_3 G_3 \coth(\xi s_3, d) = 0 \\ &D_1 G_1 \tanh(\xi s_1 d) - D_2 G_2 \tanh(\xi s_2 d) + D_3 G_3 \tanh(\xi s_3, d) = 0 \end{aligned} \tag{14}$$

where

$$D_j = iF - m_j s_j - l_j$$

$$G_1 = -Y_3 Z_2, \qquad G_2 = -Y_3 Z_1, \qquad G_3 = Y_1 Z_2 - Y_2 Z_1$$

$$Y_j = im_j - s_j, \qquad Z_j = -l_j s_j, \qquad j = 1, 2, 3$$

m_j and l_j are given in Eqs. (8).

These are the periodic equations, which correspond to the symmetric and antisymmetric motion of the plate with respect to the medial plane $z = 0$. It can be shown that first Eq. (14) corresponds to the symmetric motion, while the second corresponds to the antisymmetric motion.

The displacements and temperature in the symmetric motion are given by

$$\begin{aligned} u &= H_1 \cosh(\xi s_1 d) + H_2 \cosh(\xi s_2 d) + H_3 \cosh(\xi s_3 d)] \exp[i\xi(x - ct)] \\ w &= -[m_1 H_1 \sinh(\xi s_1 d) + m_2 H_2 \sinh(\xi s_2 d) \\ &\quad + m_3 H_3 \sinh(\xi s_3 d)] \exp[i\xi(x - ct)] \\ T &= [l_1 H_1 \cosh(\xi s_1 d) + l_2 H_2 \cosh(\xi s_2 d)] \exp[i\xi(x - ct)] \end{aligned} \tag{15}$$

and those in the antisymmetric motion by

$$\begin{aligned} u &= H_1 \sinh(\xi s_1 d) + H_2 \sinh(\xi s_2 d) + H_3 \sinh(\xi s_3 d)] \exp[i\xi(x - ct)] \\ w &= -[m_1 H_1 \cosh(\xi s_1 d) + m_2 H_2 \cosh(\xi s_2 d) \\ &\quad + m_3 H_3 \cosh(\xi s_3 d)] \exp[i\xi(x - ct)] \\ T &= [l_1 H_1 \sinh(\xi s_1 d) + l_2 H_2 \sinh(\xi s_2 d)] \exp[i\xi(x - ct)] \end{aligned} \tag{16}$$

where m_j $(j = 1, 2, 3)$ and l_k $(k = 1, 2)$ are given in Eqs. (8).

If $\tau = \tau_G \neq 0$, Eqs.(14) become the frequency equations in the LS theory of generalized thermoelasticity, which have been discussed by Verma and Hasebe [20]. Massalas and Kalpakidis [19] have derived and discussed the frequency equation for a very thin thermally insulated isotropic plate in the context of the LS theory, mixed boundary conditions, and isothermal and insulated edges. The discussion of the transcendental Eqs. (14), in general, is difficult; we, therefore, consider the results for some limiting cases.

4. SYMMETRIC MODES

For wavelengths long compared with the thickness $2d$ of the plate, ξd is small, and consequently, ξds_1, ξds_2, and ξds_3 may be assumed small as long as c is finite. Hence, the hyperbolic functions can be replaced by their arguments and the first Eq. (14) reduces to

$$(s_1^2 - s_2^2)[(1 + s_3^2)^2 \{s_1^2 + s_2^2 + c^2 - 1\} - 4s_1^2 s_2^2] = 0 \tag{17}$$

where

$$\begin{aligned} s_1^2 + s_2^2 &= -[(c^2 - 2) + (\tau c^2 + \varepsilon_1 \tau_G c^2)] \\ s_1^2 s_2^2 &= [1 - (\tau c^2 + \varepsilon_1 \tau_G c^2) + c^4 \tau - c^2] \end{aligned} \tag{18}$$

Hence, either

$$(s_1^2 - s_2^2) = 0$$

or

$$[(1 + s_3^2)^2 \{s_1^2 + s_2^2 + c^2 - 1\} - 4s_1^2 s_2^2] = 0 \tag{19}$$

If $s_1^2 = s_2^2$, the form of the original solution assumed, cannot satisfy the boundary conditions. Hence, Eq. (19) holds. Substituting the expressions for $s_1^2 + s_2^2$ and $s_1^2 s_2^2$ from Eq. (18) and $s_3^2 = 1 - (c^2/c_2)$ into Eq. (19), we obtain

$$\left[2 - \frac{c^2}{c_2}\right]^2 [1 - c^2(\tau + \varepsilon_1 \tau_G)] = 4[(c^2\tau - 1)(c^2 - 1) - \varepsilon_1 c^2 \tau_G] \tag{20}$$

This equation gives the phase velocity of long compressional or plate waves c_p [normalized form by dividing $\sqrt{(\lambda + 2\mu)/\rho} = v_1$] according to the generalized theory of thermoelasticity. For an aluminum plate, for which the physical data are given in Section 8, the velocity of plate waves is found to be $c_p = 0.552$ (nondimensional).

The thermoelastic coupling factor ε_1 is usually small [4]. If we neglect this factor, then Eq. (20) reduces to

$$c^2 = 4\beta^2 \left(1 - \frac{\beta^2}{\alpha^2}\right) \tag{21}$$

which agrees with Ewing et al. [22].

For the case of very *short wavelengths* and c, with s_1, s_2, and s_3 real, ξd is large and the hyperbolic functions tend to unity, the first frequency Eq. (14) becomes

$$(s_1 - s_2)[-(1+s_3^2)^2 \{s_1^2 + s_1^2 + s_1 s_2 + c^2 - 1\} + 4 s_1 s_2 s_3 (s_1 + s_2)] = 0 \tag{22}$$

Evidently $(s_1 - s_2)$ is a factor. Factorizing Eq. (22), we obtain

$$[-(1+s_3^2)^2 \{s_1^2 + s_1^2 + s_1 s_2 + c^2 - 1\} + 4 s_1 s_2 s_3 (s_1 + s_2)] = 0 \tag{23}$$

Equation (23) can be identified as the phase velocity equation for Rayleigh waves in isotropic half-space. This is in agreement with the corresponding result of Nayfeh and Nasser [24]. For an aluminum alloy plate, Rayleigh waves are found to propagate with a velocity $c_R = 0.384$ (nondimensional).

If we set $\tau_1 = \tau_0 = 0$ (the case of coupled thermoelasticity), when there is no thermal relaxation time, expressions for τ and τ_G defined after Eq. (6) reduce to $\tau = \tau_G = i/\xi c$. Proceeding along the same lines as in the previous section, we obtain an equation similar to Eq. (23), with s_1 and s_2 given by

$$\begin{aligned} s_1^2 + s_2^2 &= -[c^2 - 2 + ci\xi^{-1}(1+\varepsilon_1)] \\ s_1^2 s_2^2 &= [(ci\xi^{-1} - 1)(c^2 - 1) - ci\xi^{-1}\varepsilon_1] \end{aligned} \tag{24}$$

and s_3 is given in Eq. (9).

Substituting the value of $s_1^2 + s_2^2$, $s_1^2 s_2^2$ from Eq. (24), with the condition ω $(=\xi c) \ll 1$, in Eq. (23), after some algebraic manipulations, Eq. (23) reduces to

$$(1+\varepsilon_1)\left[2 - \frac{c^2}{c_2}\right]^4 = 16((1+\varepsilon_1) - c^2)\left(1 - \frac{c^2}{c_2}\right) \tag{25}$$

which agrees with Ref. 23.

In the classical case when the strain and thermal fields are uncoupled from each other, then the coupling constant ε_1 is identically zero, and Eq. (25) reduces to

$$\left[2-\frac{c^2}{c_2}\right]^4 = 16(1-c^2)\left(1-\frac{c^2}{c_2}\right) \tag{26}$$

This is in agreement with the corresponding result of Nayfeh and Nasser [24].

If we set $\tau_1 = \tau_0$, then it is seen that the results obtained in the GL theory reduce to the results in the LS theory of generalized thermoelasticity.

5. ANTISYMMETRIC MODES

For wavelengths long compared to the thickness of the plate, with s_1, s_2, and s_3 real, we may replace the hyperbolic functions by the approximation

$$\tanh \cong x - x^3/3 \tag{27}$$

After some algebraic transformation and reductions, and neglecting the quantities of $O[\xi d]^3$ in the second Eq. (14), we obtain

$$\frac{c^2}{c_2} - \frac{4\xi^2 d^2}{3}\left[(c_2-1)\left(1+\frac{c^2}{c_2}\right) - \frac{c^2}{4c_2}(c^2-1)\right] \tag{28}$$

This is the dispersion equation for long flexural waves, and it can be seen that the phase velocity tends to zero as the wavelength increases to infinity.

For wavelengths short compared with the thickness of the plate, that is, $\xi d \to \infty$, and c such that s_1, s_2, and s_3 are real, the second Eq. (14) reduces to the Rayleigh Eq. (23), and the propagation degenerates to the Rayleigh waves associated with both free surfaces of the plate in generalized thermoelasticity.

If we take $\tau_1 = \tau_0$, then all the results in the GL theory reduce to the corresponding results in the LS theory of generalized thermoelasticity. If $\tau_0 = \tau_1 = 0$, then the results obtained here reduce to the coupled thermoelasticity (here in the dimensionless form).

6. THERMOELASTICITY WITHOUT ENERGY DISSIPATION

The fundamental equations for such a medium, with heat sources and body forces absent, in the context of generalized thermoelasticity developed by Green and Naghdi [5], are given by

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)+(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x\,\partial z}\right)-\beta\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}$$

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)+(\lambda+\mu)\left(\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 u}{\partial x\,\partial z}\right)-\beta\frac{\partial T}{\partial z}=\rho\frac{\partial^2 w}{\partial t^2}$$

$$K^*\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right)-\rho C\frac{\partial^2 T}{\partial t^2}=\beta T_0\left(\frac{\partial^3 u}{\partial x\,\partial t^2}+\frac{\partial^3 w}{\partial z\,\partial t^2}\right) \tag{29}$$

$$\beta=(3\lambda+2\mu)\,\alpha_t$$

where K^* is a characteristic constant of the medium, and ρ and C are, respectively, the mass density and specific heat at constant strain. All other symbols and notations used in Eqs. (29) have the same meaning as defined in Eqs. (1).

$$C_1^2=\frac{\lambda+2\mu}{\rho},\qquad C_2^2=\frac{\mu}{\rho},\qquad C_3^2=\frac{K^*}{\rho C}=\omega^{*-1}C_1^2 \tag{30}$$

$\omega^*=C(\lambda+2\mu)/K^*$ is the characteristic frequency of the medium.

Substituting u, w, and T from Eqs. (5) into Eqs. (29) and using Eq. (30), we obtain

$$(C_2^2D^2-C_1^2\xi^2+\xi^2c^2)\,f+i\xi(C_1^2-C_2^2)\,Dg-iC_1^2\xi h=0$$

$$i\xi(C_1^2-C_2^2)\,Df+(C_1^2D^2-C_2\xi^2+\xi^2c^2)\,g-C_1^2\,Dh=0 \tag{31}$$

$$i\xi^3\varepsilon_1c^2f+\varepsilon_1\xi^2c^2\,Dg+(C_3^2(D^2-\xi^2)+\xi^2c^2)\,h=0$$

where D, ε_1, and ξ have the same meaning as defined in the previous sections.

The solution to Eqs. (31) is again of the form in Eq. (7), where

$$m_j=is_j,\qquad m_3=0,\quad j=1,2$$

$$l_j=\frac{1}{iC_1^2}\,[C_1^2s_j^2-C_1^2+c^2],\qquad l_3=0,\quad j=1,2 \tag{32}$$

s_1^2 and s_2^2 are the roots of the equation

$$s^4+As^2+B=0 \tag{33}$$

where

$$A = \frac{[(1+\varepsilon_1)\,C_1^2 + C_3^2)\,c^2 - C_1^2 C_3^2)]}{\Delta}$$

$$B = \frac{[c^4 - \{(1+\varepsilon_1)\,C_1^2 + C_3^2\}\,c^2 - C_1^2 C_3^2]}{\Delta}$$

$$\Delta = C_1^2 C_3^2$$

and

$$s_3^2 = 1 - \frac{c^2}{C_2} \tag{34}$$

s_1^2 and s_2^2 correspond to the coupled longitudinal and thermal waves, whereas s_3^2 corresponds to the transverse wave in thermoelasticity without energy dissipation.

When there is no coupling, i.e., $\varepsilon_1 = 0$, then Eq. (33) further factorizes to

$$s_1^2 = \frac{c^2}{C_1^2} - 1, \qquad \text{and} \qquad s_2^2 = \frac{c^2}{C_3^2} - 1 \tag{35}$$

Thus, we see that s_1^2 and s_2^2 correspond to purely elastic and thermal waves, respectively.

Invoking the boundary conditions in Eq. (12) on the plate outer boundaries, frequency equations, displacements, and temperatures for the symmetric and antisymmetric wave modes, corresponding to Eqs. (14)–(16) of the LS and GL theories, are obtained in the GN theory, with s_1^2, s_2^2, m_j ($j = 1, 2, 3$), and l_k ($k = 1, 2$) being given in Eqs. (33) and (34), respectively.

Discussion of the frequency equations obtained for the linear theory of thermoelasticity without energy dissipation is, in general, difficult; we, therefore, consider the results for some limiting cases.

6.1. Symmetric Modes

For wavelengths long compared with the thickness $2d$ of the plate, the first Eq. (14) in the GN theory reduces to

$$\begin{aligned}\left[2 - \frac{c^2}{C_2^2}\right]^2 &[C_1^2 C_3^2 - c^2\{C_3^2 + (1+\varepsilon_1)\,C_1^2\} + c^2 C_1^2 C_3^2] \\ &= 4[(c^4 - c^2\{C_3^2 + (1+\varepsilon_1)\,C_1^2\} + C_1^2 C_3^2\end{aligned} \tag{36}$$

This equation gives the phase velocity of long compressional or plate waves in the linear theory of thermoelasticity without energy dissipation.

For very short wavelengths, and c, s_1, s_2, and s_3 are real, ξd is large, and the hyperbolic functions tend to unity. Then in the GN theory, we obtain the frequency equations analogous to Eqs. (22) and (23) in the context of the LS and GL theories, with s_1 and s_2 given in Eq. (33).

In the special case where the strain and thermal fields are uncoupled from each other, the coupling constant ε_1 is identically zero, and Eq. (23) in the GN theory reduces to

$$\left[2-\frac{c^2}{C_2^2}\right]^4 = 16(1-c^2)\left(1-\frac{c^2}{C_2^2}\right) \tag{37}$$

which is again of the same form as Eq. (26) in the LS and GL theories, with C_2^2 given in Eq. (30) for the GN theory.

6.2. Antisymmetric Modes

For wavelengths long compared with the thickness of the plate, s_1, s_2, and s_3 are real. We may replace the hyperbolic functions with the approximation of Eq. (27). Then Eq. (28) reduces to

$$\frac{c^2}{C_2^2} - \frac{4\xi^2 d^2}{3}\left[(C_2^2-1)\left(1+\frac{c^2}{C_2^2}\right) - \frac{c^2}{4C_2^2}(c^2-1)\right] \tag{38}$$

which is of the same form as Eq. (28) in the LS and GL theories, with C_2^2 as in Eq. (30) for the GN theory.

This is the dispersion equation for long flexural waves, and it can be seen that the phase velocity tends to zero as the wavelength increases to infinity in the linear theory of thermoelasticity without energy dissipation.

For wavelengths short compared with the thickness of the plate, that is, $\xi d \to \infty$, and c such that s_1, s_2, and s_3 are real, Eqs. (14) reduce to the Rayleigh Eq. (23), and the propagation degenerates to the Rayleigh waves associated with both free surfaces of the plate in this theory.

7. DISCUSSION AND CONCLUSIONS

As discussed earlier, the frequency Eqs. (14) are complex so that these transcendental equations enable us to evaluate not only the phase velocity, but also the thermoelastic energy dissipation for the propagation of thermoelastic waves in an infinite plate. In general, the waves are dispersive and dissipate energy according to the LS and GL theories of generalized

Table I. Physical Properties of an Aluminum Alloy

Young's modulus E	72.6 GPa
Poisson ratio ν	0.33
Density ρ	$2800\ \mathrm{kg \cdot m^{-3}}$
Specific heat C_e	$960\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}}$
Thermal diffusivity k_1	$7 \times 10^{-5}\ \mathrm{m^2 \cdot s^{-1}}$
Expansion coefficient α_t	$2.35 \times 10^{-5}\ \mathrm{K^{-1}}$
Initial temperature T_0	293 K

thermoelasticity. To find the manner in which the long and short wavelength limits are connected requires numerical solution of Eqs. (14). Moreover, for values of c which make s_1, s_2, and s_3 imaginary, the hyperbolic functions become periodic and so an infinite number of higher modes exists. Computations for the symmetric and antisymmetric modes have been carried out for an aluminum alloy plate whose physical data are given in Table I.

The phase and group velocities [c and $U = c + \xi(dc/d\xi)$, respectively] and dispersion curves are plotted as a function of the wave number assuming that the thickness $2d$ of the plate is fixed. These curves have been calculated from an expression based on the dispersion relation in Eqs. (14), which are decoupled characteristic equations corresponding to symmetric and antisymmetric modes of vibrations according to the LS and GL theories of generalized thermoelasticity.

The additional new mode to those already observed in purely elastic materials is the quasi-thermal mode (T-mode). Dispersion curves for symmetric and antisymmetric modes in LS and GL theories of generalized thermoelasticity are shown in Figs. 1 and 2. The various modes approach each other and then merge as the wave number increases, where the phase and group velocities tend toward the Rayleigh surface wave speed. It is observed that wave modes are more affected at the zero wave-number limits, due to the thermomechanical effects, which supports the idea that second sound effects are short- lived. This clearly demonstrates the difference between the coupled and the generalized theory of thermoelasticity. In the first mode of symmetric vibration, the phase velocity decreases monotonically with increasing values of wave number, from c_p (plate velocity) at $\xi = 0$ to c_R (Rayleigh surface wave speed) at $\xi = \infty$. The group velocity has the same asymptotic limits but has a minimum. In the second mode, the phase velocity is higher than the horizontally polarized shear [$c_H = (\mu/\rho)^{1/2}$] or SH wave in the plate. Again, $c \to \infty$ and $U \to 0$ as $\xi \to 0$, and as $\xi \to \infty$, c and $U \to$ the horizontally polarized shear [$c_H = (\mu/\rho)^{1/2}$] or SH wave in the plate. Both the maximum and the minimum values of group velocity are associated with this mode at intermediate wave numbers.

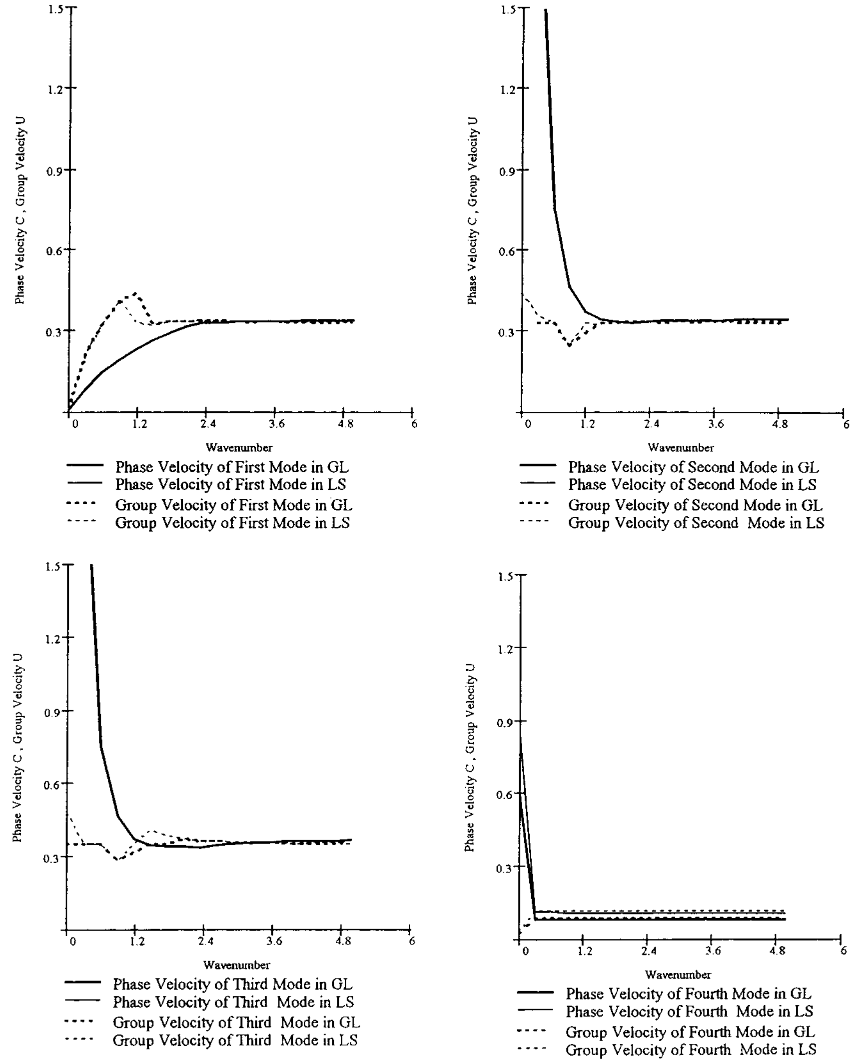

Fig. 1. Dispersion curves in the LS and GL theories of generalized thermoelasticity for antisymmetric modes.

Similar relations between phase and group velocity for higher modes are demonstrated in the dispersion curves in Fig. 1.

In the first mode of antisymmetric vibration (Fig. 2) the phase velocity increases monotonically with increasing values of wave number ξ from $c=0$ at $\xi=0$ to $c=c_R$ at $\xi=\infty$. As $\xi\to 0$, $U\to 0$, which is characteristic

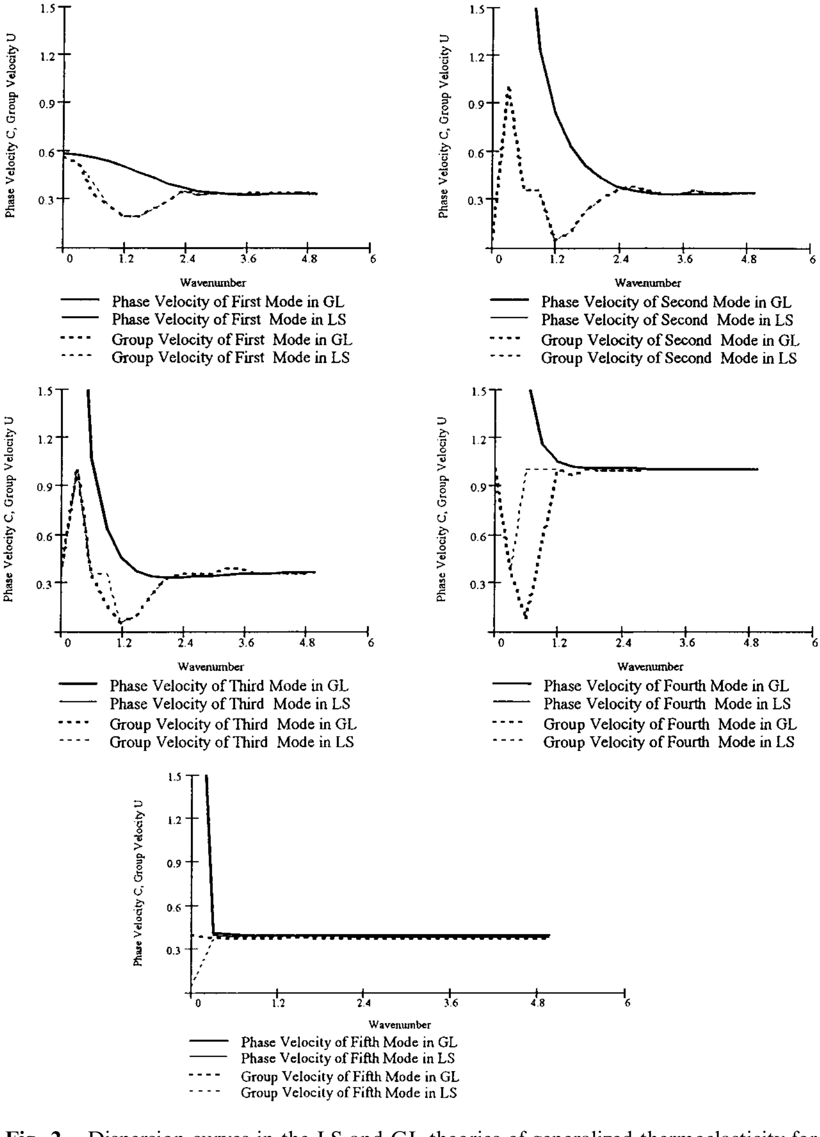

Fig. 2. Dispersion curves in the LS and GL theories of generalized thermoelasticity for symmetric modes.

of flexural waves, and as $\xi \to \infty$, c and $U \to c_R$ in the plate. The maximum value of group velocity is equal to the horizontally polarized shear $[c_H = (\mu/\rho)^{1/2}]$ or SH wave in the plate. The results obtained for flexural mode (first mode) are in agreement with corresponding results obtained by Ewing et al. [22] (see their Figs. 6–18). Dispersion curves (antisymmetric) for phase and group velocities for higher modes in the LS and GL theories are shown in Fig. 2. The maximum value of the phase and group velocity curves for the fourth mode (antisymmetric), Fig. 2, and fifth mode (symmetric), Fig. 1, approach the c axis at a low wave number, at such large values that these are multiplied by 10^{-4} so that they can be seen in the figures. Although the waves in the context of the GL theory are subject to

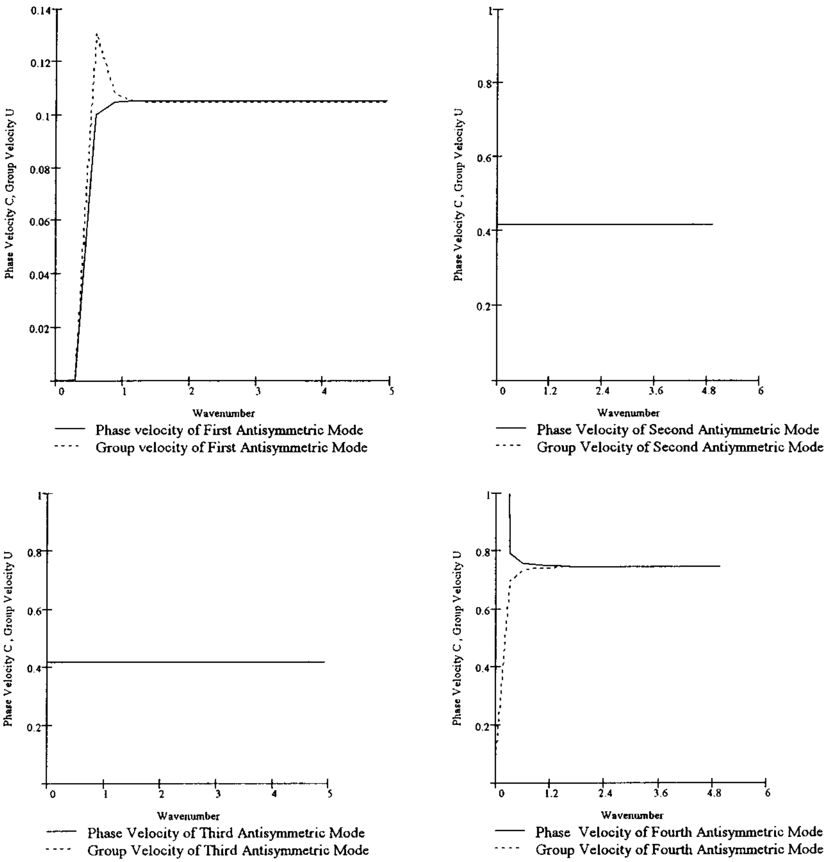

Fig. 3. Dispersion curves for antisymmetric wave modes in the GN theory of thermoelasticity.

stronger modifications than those in the LS theory; in general, both theories lead to similar conclusions and results.

Similar dispersion curves for antisymmetric and symmetric modes according to the GN theory of generalized thermoelasticity for an aluminum plate are shown in Figs. 3 and 4. It has been found that phase velocity is equal to group velocity, i.e., $c = U$ for the second and third modes (antisymmetric) and third and fourth modes (symmetric), and therefore, these modes show no dispersion in the GN theory.

As the exponential term in Eqs. (5) is a function of x and t, thermoelastic waves may undergo spatial attenuation in the direction of propagation. Specifically, for a given value of wave number ξ, c must be a complex number of the form $c = c_{re} + ic_{im}$ in both the LS and the GL theories of

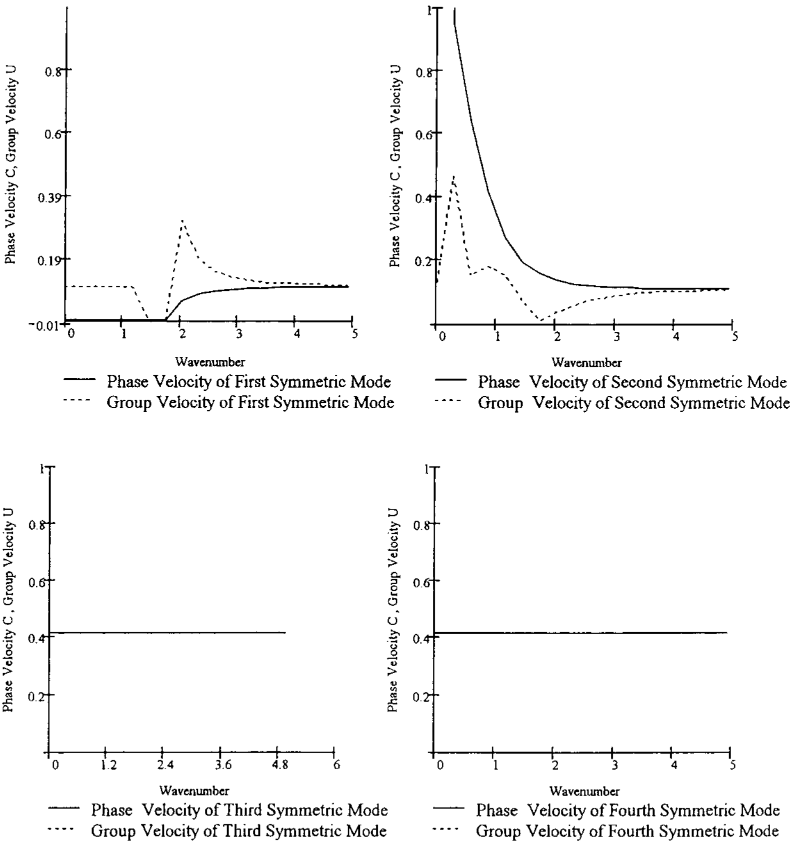

Fig. 4. Dispersion curves for symmetric wave modes in the GN theory of thermoelasticity.

generalized thermoelasticity. In this case, from the exponential term in Eqs. (5), we can see that $\mathrm{Re}(c) = c_{re}$ gives the phase velocity and the imaginary part, $\mathrm{Im}(c) = c_{im}$ of c gives the attenuation coefficient for waves. Figures 5 and 6 show the wave-number dependence of the attenuation coefficient for antisymmetric and symmetric waves in a thermoelastic aluminum

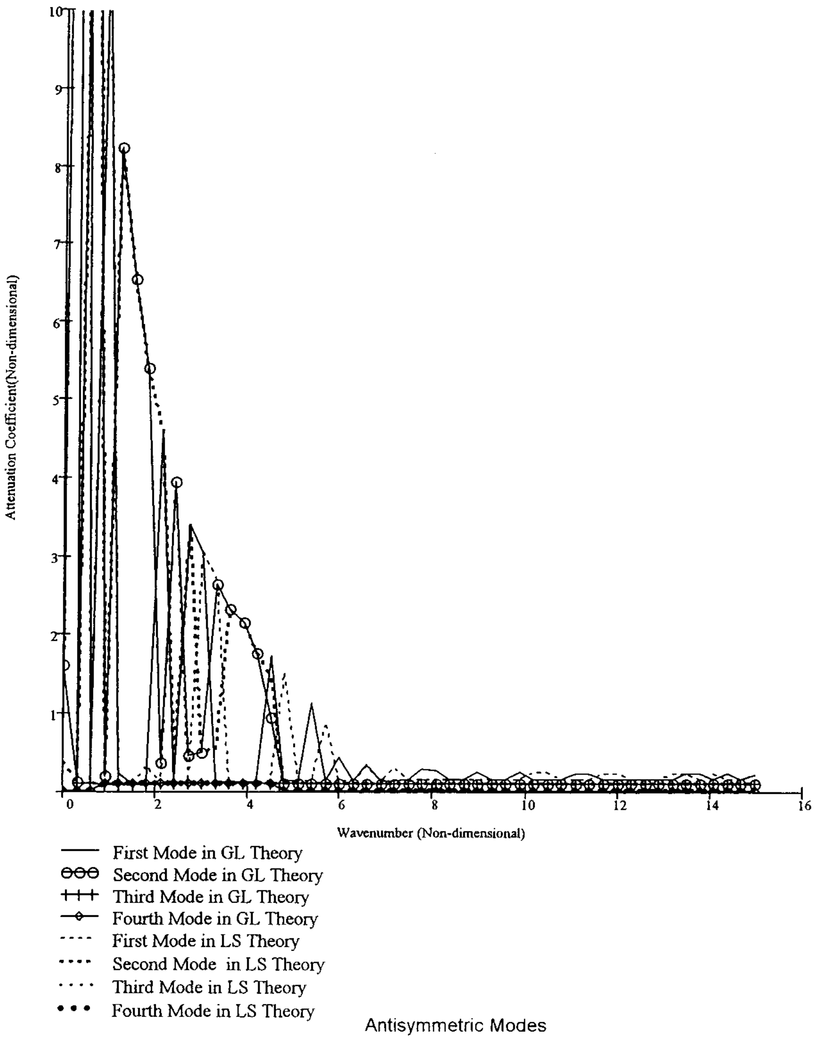

Fig. 5. Wave-number dependence of the thermoelastic attenuation constant in the GL and LS theories of generalized thermoelasticity for antisymmetric modes.

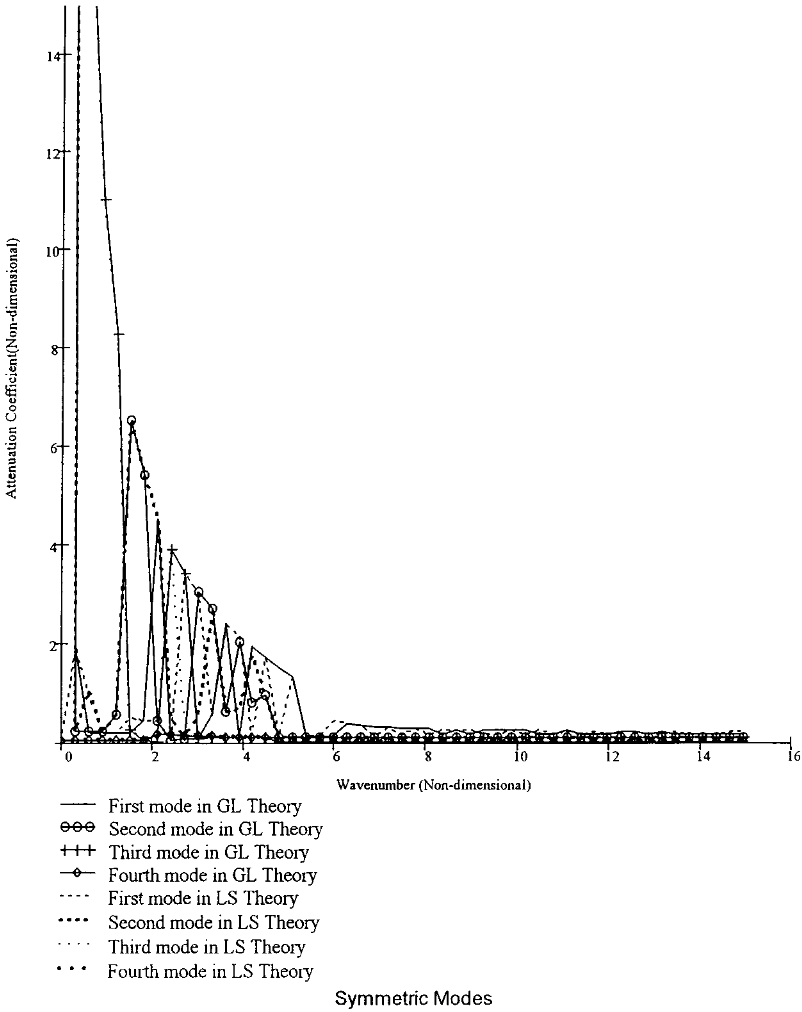

Fig. 6. Wave-number dependence of the thermoelastic attenuation constant in the GL and LS theories of generalized thermoelasticity for symmetric modes.

alloy plate. Each of the curves in these figures corresponds to one of the branches in Figs. 1 and 2 (up to the first four modes), respectively.

Furthermore, the solutions obtained in the GN theory show that there exist symmetric and antisymmetric modes of coupled (thermal and elastic wave modes) waves, without any attenuation. That this is not the case in the LS and GL theories is an interesting feature inherent in the GN theory. In the LS and GL theories, waves experience attenuation, and the attenuation factors decay exponentially [25, 26]. It has also been observed that the predictions of the GN theory are qualitatively similar to those of the LS and GL theories.

When the thermal relaxation time $\tau_0 \rightarrow 0$, then the results obtained in the analysis reduce to the conventional coupled theory of thermoelasticity. When the coupling constant ε_1 is identically equal to zero, the strain and thermal fields are uncoupled from each other. In this case the results can be obtained from the uncoupled theory of thermoelasticity.

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